A ONE-PARAMETER SOLUTION OF THE LAMINAR BOUNDARY-LAYER EQUATIONS IN A GAS WITH ARBITRARY OUTER-FLOW VELOCITY AND ARBITRARY TEMPERATURE DIFFERENCE

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The author presents the methods and results of numerical integration of universal laminar boundary-layer equations (in the one-parameter approximation) for a gas flow with large velocities, a Prandtl number equal to unity, and a linear relation between the dynamic viscosity and temperature.

1. THE UNIVERSAL EQUATIONS IN A ONE-PARAMETER APPROXIMATION

The system of universal equations for the laminar boundary layer in a high-speed gas flow with arbitrary outer velocity $u_l(x)$, arbitrary ratio of body to free-stream temperature T_w/T_∞, and Prandtl number equal to unity is, in the one-parameter approximation [1, 2],

$$\frac{\partial^3\Phi}{\partial\xi^3}+\frac{F+2f_1}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\xi^2}+\frac{f_1}{B^2}\left[1+S-\left(\frac{\partial\Phi}{\partial\xi}\right)^2\right]=$$
$$=\frac{1}{B^2}Ff_1\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi\partial f_1}-\frac{\partial\Phi}{\partial f_1}\frac{\partial^2\Phi}{\partial\xi^2}\right);$$
$$\frac{\partial^2 S}{\partial\xi^2}+\frac{F+2f_1}{2B^2}\Phi\frac{\partial S}{\partial\xi}=$$
$$=\frac{1}{B^2}Ff_1\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial S}{\partial f_1}-\frac{\partial\Phi}{\partial f_1}\frac{\partial S}{\partial\xi}\right);$$
$$\Phi=\frac{\partial\Phi}{\partial\xi}=0,\ S=S_w\quad\text{at}\quad\xi=0;$$
$$\frac{\partial\Phi}{\partial\xi}\to1,\quad S\to0\quad\text{at}\quad\xi\to\infty;$$
$$\Phi=\Phi_0(\xi),\quad S=S_0(\xi)\ \text{when}\ f_1=0.\qquad(1)$$

The unknown functions $\Phi(\xi,f_1)$ and $S(\xi,f_1)$ are the "reduced" stream function and heat function, defined as

$$\Phi=B\Psi/(V_l\Delta^{**}),\quad S=h_0/h_{0l}-1.\qquad(2)$$

The independent variables in (1) are

$$\xi=BY/\Delta^{**}\qquad(3)$$

and the basic boundary-layer shape parameter

$$f_1=(dV_l/dX)\Delta^{**2}/\nu_{0l}.\qquad(4)$$

Here we use the following notation: $\Delta^{**}=\int\limits_0^\infty\frac{V}{V_l}\left(1-\frac{V}{V_l}\right)dY$ —transformed momentum thickness; X, Y, V—coordinates and velocity transformed by the Dorodnitsyn-Stewartson transformation [3]; V_l—transformed outer-flow velocity; Ψ—transformed stream function; h_0 and h_{0l}—total enthalpy inside the boundary layer and of the outer flow, respectively; ν_{0l}—kinematic viscosity corresponding to stagnation conditions in the outer flow.

The normalization constant B is determined from the condition that for $f_1=0$ system (1) should reduce to the corresponding system of equations for a semi-infinite plate in parallel flow (the dot denotes differentiation with respect to ξ):

$$\dddot{\Phi}_0+\Phi_0\ddot{\Phi}_0=0;$$
$$\ddot{S}_0+\Phi_0\dot{S}_0=0;$$
$$\Phi_0=0,\quad\dot{\Phi}_0=0,\quad S_0=S_w\quad\text{at}\quad\xi=0;$$
$$\dot{\Phi}_0\to1,\quad S_0\to0\quad\text{at}\quad\xi\to\infty.$$

Taking into account that in the general case ($\Delta^*=\int\limits_0^\infty\left(1+S-\frac{V}{V_l}\right)dY$ is the transformed displacement thickness)

$$F=2[\zeta-(2+H)f_1],\quad\zeta=B\left(\frac{\partial^2\Phi}{\partial\xi^2}\right)_{\xi=0},$$
$$H=\frac{\Delta^*}{\Delta^{**}}=\frac{1}{B}\int\limits_0^\infty\left(1+S-\frac{\partial\Phi}{\partial\xi}\right)d\xi,$$

we have

$$B=\ddot{\Phi}_0(0).$$

2. METHOD OF INTEGRATION OF SYSTEM (1)

The system of equations (1) was integrated on the BESM-2 computer of the Leningrad Computing Center of the Academy of Sciences USSR for the following values of the parameter S_ω: −0.6; −0.4; −0.2; 0.2; 0.4. The method of integration of the universal equation for the case $S_w=0$ has been discussed by Simuni and Terent'ev [4]. Introducing the notation

$$u=\frac{\partial\Phi}{\partial\xi},\quad v=-\frac{\partial\Phi}{\partial f_1},$$

we rewrite system (1) in the form

$$\frac{\partial^2u}{\partial\xi^2}+\frac{F+2f_1}{2B^2}\Phi\frac{\partial u}{\partial\xi}+\frac{f_1}{B^2}(1+S-u^2)=$$

$$= \frac{1}{B^2} Ff_1 \left(u \frac{\partial u}{\partial f_1} + v \frac{\partial u}{\partial \xi} \right);$$

$$\frac{\partial u}{\partial f_1} + \frac{\partial v}{\partial \xi} = 0;$$

$$\frac{\partial^2 S}{\partial \xi^2} + \frac{F + 2f_1}{2B^2} \Phi \frac{\partial S}{\partial \xi} = \frac{1}{B^2} Ff_1 \left(u \frac{\partial S}{\partial f_1} + v \frac{\partial S}{\partial \xi} \right);$$

$$\Phi = u = v = 0, \quad S = S_w \quad \text{at} \quad \xi = 0;$$

$$u \to 1, \quad S \to 0 \quad \text{at} \quad \xi \to \infty;$$

$$\Phi = \Phi_0(\xi), \quad u = u_0(\xi), \quad v = v_0(\xi),$$

$$S = S_0(\xi) \text{ when } f_1 = 0. \qquad (5)$$

System (5) was approximated by the finite-difference scheme (the indices i and k refer to fixed values of f_1 and ξ, respectively)

$$\frac{u_{i+1,k+1} - 2u_{i+1,k} + u_{i+1,k-1}}{\Delta\xi^2} +$$

$$+ \left(\frac{F + 2f_1}{2B^2} \right)_i \Phi_{i,k} \frac{u_{i,k+1} - u_{i,k-1}}{2\Delta\xi} + \left(\frac{f_1}{B^2} \right)_i [S_{i,k} -$$

$$- (u_{i+1,k} - 1)(u_{i,k} + 1)] = \frac{1}{B^2} (Ff_1)_i \left(u_{i,k} \frac{u_{i+1,k} - u_{i,k}}{\Delta f_1} + \right.$$

$$\left. + v_{i,k} \frac{u_{i,k+1} - u_{i,k-1}}{2\Delta\xi} \right);$$

$$v_{i+1,k+1} = v_{i+1,k} - \frac{\Delta\xi}{2} \left(\frac{u_{i+1,k+1} - u_{i,k+1}}{\Delta f_1} + \frac{u_{i+1,k} - u_{i,k}}{\Delta f_1} \right);$$

$$\Phi_{i+1,k+1} = \Phi_{i+1,k} + \frac{\Delta\xi}{2} (u_{i+1,k+1} + u_{i+1,k});$$

$$\frac{S_{i+1,k+1} - 2S_{i+1,k} + S_{i+1,k-1}}{\Delta\xi^2} +$$

$$+ \left(\frac{F + 2f_1}{2B^2} \right)_i \Phi_{i+1,k} \frac{S_{i,k+1} - S_{i,k-1}}{2\Delta\xi} =$$

$$= \frac{1}{B^2} (Ff_1)_i \left(u_{i+1,k} \frac{S_{i+1,k} - S_{i,k}}{\Delta f_1} + v_{i+1,k} \frac{S_{i,k+1} - S_{i,k-1}}{2\Delta\xi} \right);$$

$$\Phi_{i,0} = u_{i,0} = v_{i,0} = 0; \quad S_{i,0} = S_w \quad \text{at} \quad \xi = 0;$$

$$u_{i,\infty} = 1; \quad S_{i,\infty} = 0 \quad \text{at} \quad \xi = \infty;$$

$$\Phi_{0,k} = \Phi_0(\xi); \quad u_{0,k} = u_0(\xi); \quad v_{0,k} = v_0(\xi);$$

$$S_{0,k} = S_0(\xi) \text{ when } f_1 = 0. \qquad (6)$$

Here $\Delta\xi$ and Δf_1 are the steps in the direction of ξ and f_1, respectively.

System (6) was solved by the double sweep method [5]. In the f_1 direction the calculation extended up to the separation point in the expansion region and up to the forward stagnation point in the compression region. The outer boundary of the boundary layer was assumed to be at $\xi = 6$. The step size of $\Delta\xi$ was 0.05. The step size in the f_1 direction was chosen so that the differences between the values of the basic boundary-layer variables calculated with full steps and with half-size steps was less than one or two units in the fourth significant figure. Thus the step size varied between $\Delta f_1 = 5 \cdot 10^{-5}$ and $\Delta f_1 = 0.3125 \cdot 10^{-5}$.

3. NUMERICAL RESULTS

Figure 1 shows curves of the dimensionless velocity $V/V_l = u/u_l = \partial\Phi/\partial\xi$ and the heat function S as functions of ξ for several values of the shape factor f_1 and $S_w = 0.4$. Figs. 2–4 show curves of the friction parameter ζ, the functions H and F, the function ε, which characterizes the deflection of the F curve from its tangent at the point $f_1 = 0$, and the reduced heat-transfer coefficient $\zeta^* = B(\partial S/\partial\xi)_{\xi=0}$, as functions of the shape factor f_1 for S_w equal to −0.4 and 0.4.

Figure 2 shows that in the compression region of the boundary layer the friction on a cooled wall is less than on a heated wall. In the expansion region the effect is opposite. This effect is due to the change of density of the gas inside the layer due to the wall temperature.

The effect of the parameter S_w on some properties of the boundary layer at the separation point ($f_{1\text{sep}}$ and ζ^*_{sep}) can be seen from Fig. 5. An increase of S_w (i.e. a transition from cooling to heating) results in an earlier separation of the boundary layer. Figure 5 also shows that an increase in the absolute value of S_w results in an increase of the absolute value of the reduced heat-transfer coefficient at the separation point, this effect being stronger in the case of heating than in the case of cooling.

In addition to the numerical solution of system (1), this system was also represented in terms of a series of the shape factors $f_k = V_l^{k-1} V_l^{(k)} z^{**k}$ $(k = 1, 2, \ldots;$ $z^{**} = \Delta^{**2}/\nu_{0l})$. Starting from the one-parameter solution ($\zeta^{(1)}$, $H^{(1)}$, $F^{(1)}$, and $\zeta^{*(1)}$), obtained numerically by means of a computer, the effect of the subsequent parameters is found to be (we give here only the expansions for $S_w = -0.4$ and $S_w = 0.4$):

for $S_w = -0.4$

$$\zeta = \zeta^{(1)} - 0.213f_2 - 0.013f_1f_2 + 0.068f_3 + \ldots;$$

$$H = H^{(1)} + 1.720f_2 + 7.418f_1f_2 - 0.822f_3 + \ldots;$$

$$F = F^{(1)} - 0.426f_2 - 3.466f_1f_2 + 0.136f_3 + \ldots;$$

$$\zeta^* = \zeta^{*(1)} + 0.066f_2 + 0.894f_1f_2 - 0.044f_3 + \ldots;$$

for $S_w = 0.4$

$$\zeta = \zeta^{(1)} - 0.385f_2 + \ldots;$$

$$H = H^{(1)} + 0.806f_2 + \ldots;$$

$$F = F^{(1)} - 0.770f_2 + \ldots;$$

$$\zeta^* = \zeta^{*(1)} - 0.114f_2 + \ldots.$$

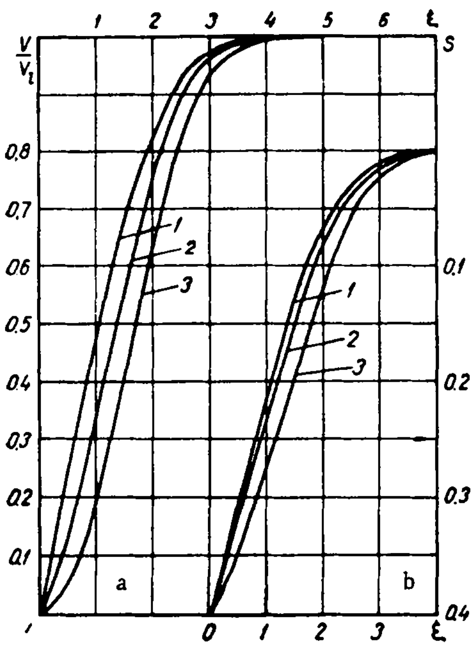

Fig. 1. Variation of the dimensionless velocity (a) and heat function (b) across the boundary layer in the case $S_w = 0.4$: 1) $f_1 = 0$; 2) $f_1 = -0.05$; 3) $f_1 = -0.0646$.

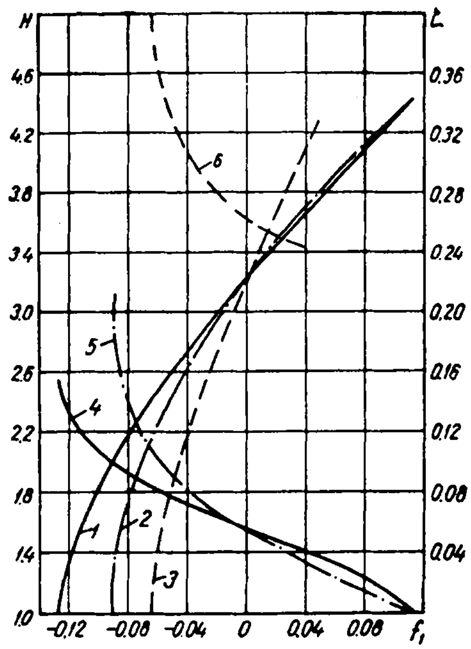

Fig. 2. Variation of the friction parameter ζ and the parameter H with the shape factor f_1: 1 and 2) ζ and ζ_{CR} for $S_w = -0.4$; 3) ζ for $S_w = 0.4$; 4 and 5) H and H_{CR} for $S_w = -0.4$; 6) H for $S_w = 0.4$.

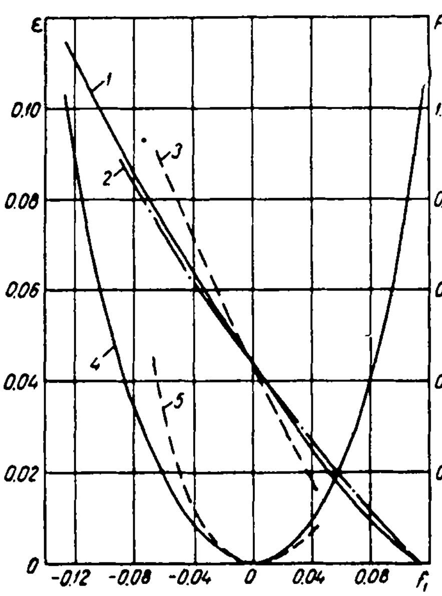

Fig. 3. Variation of F and ε with the shape factor f_1: 1 and 2) F and F_{CR} for $S_w = -0.4$; 3) F for $S_w = 0.4$; 4 and 5) ε for $S_w = -0.4$ and $S_w = 0.4$.

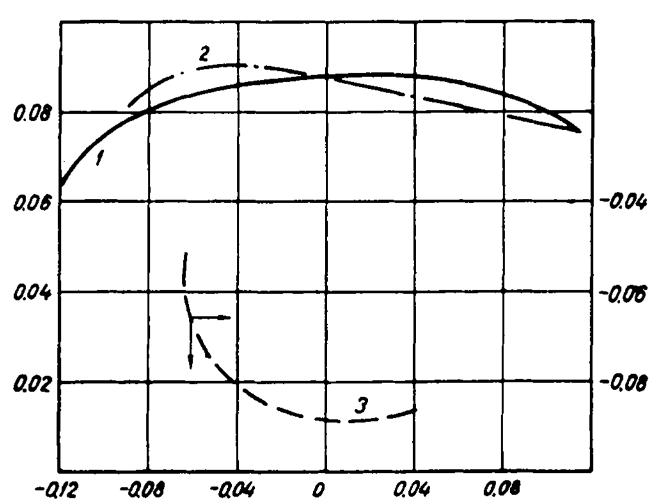

Fig. 4. Variation or the reduced heat-transfer coefficient ζ^* (on the ordinate axis) with the shape factor f_1 (abscissas): 1 and 2) ζ^* and ζ_{CR} for $S_w = 0.4$; 3) ζ^* for $S_w = 0.4$.

At the present time work is in progress on the approximate integration (in the one-parameter approximation) of the universal equations of the laminar boundary layer in a flow of homogeneous gas with Prandtl number equal to 0.72. However, even the present results for Pr = 1 are quite valuable, since it has been shown in [6, 7] that the effect of the Prandtl number on the dynamic characteristics of a boundary layer is practically insignificant.

4. COMPARISON OF THE ONE-PARAMETER METHOD OF SOLUTION OF THE BOUNDARY LAYER WITH THE METHOD OF COHEN AND RESHOTKO [8]

Assuming all derivatives with respect to f_1 in (1) as equal to zero, we obtain a system of ordinary differential equations, with f_1 as parameter. As is well known, the problem of a boundary layer in a high-speed gas flow becomes self-similar when the transformed velocity of the outer flow is given as a power function of the length coordinate (C and m are constants)

$$V_l = CX^m.$$

Introducing the expressions

$$f_1 = \beta B^2, \quad F = 2B^2(1-\beta), \quad \beta = 2m/(m+1),$$

we can write the above system of differential equations in the form

$$\dddot{\Phi} + \Phi\ddot{\Phi} = \beta(\dot{\Phi}^2 - 1 - S),$$
$$\ddot{S} + \Phi\dot{S} = 0. \qquad (7)$$

This system is equivalent to the equations which have been obtained by Stewartson [3] and integrated by Cohen and Reshotko [9]. Using the class of exact solutions, these authors have proposed an approximate method of calculation of the boundary layer [8].

From what we have said above it follows that the method of Cohen and Reshotko can give satisfactory results only in those cases in which the derivatives of the unknown functions with respect to f_1 are relatively small. This assumption holds in the compression region of the boundary layer. However, near the separation point of the boundary layer the effect of the derivatives with respect to f_1 increases (see, e.g., Fig. 1) and the Cohen-Reshotko method does not lead to satisfactory results.

Figures 2–4 show in broken lines the curves of ζ_{CR}, H_{CR}, F_{CR}, and ζ^*_{CR}, calculated by the Cohen-Reshotko method. The results obtained by this method yield lower friction values in the expansion region and premature separation.

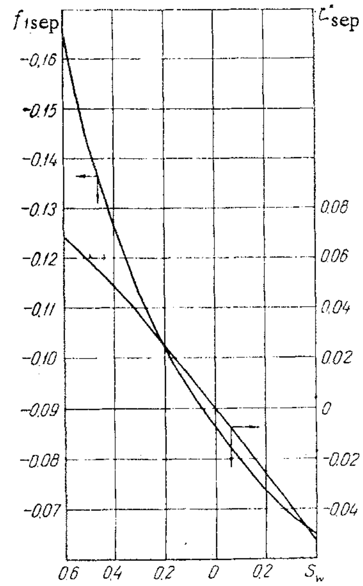

Fig. 5. The effect of the parameter S_w on the shape factor f_{1sep} and the reduced heat-transfer coefficient ζ^*_{sep} at the separation point of the boundary layer.

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